SMP 16-19 Mathematics – Revision Notes
Unit 4 – Mathematical Methods

Trigonometry

1. Pythagoras' theorem can be used in three dimensions so that $d^2 = a^2 + b^2 + c^2$.
2. Particular values of trigonometric functions:

$$\sin 45° = \frac{1}{\sqrt{2}} \quad \cos 45° = \frac{1}{\sqrt{2}} \quad \tan 45° = 1$$
$$\sin 30° = \frac{1}{2} \quad \cos 30° = \frac{\sqrt{3}}{2} \quad \tan 30° = \frac{2}{\sqrt{3}}$$
$$\sin 60° = \frac{\sqrt{3}}{2} \quad \cos 60° = \frac{1}{2} \quad \tan 60° = \sqrt{3}$$

3. The equation of a circle of radius *r* about (*a*, *b*) is given by $(x-a)^2 + (y-b)^2 = r^2$.
4. The following trigonometric identities are true, for any value of θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \sin^2\theta + \cos^2\theta = 1$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

5. An equation of the form $a\sin\theta + b\cos\theta$ can be written in the form $R\sin(\theta+\alpha)$:

$$R = \sqrt{(a^2+b^2)}; \quad \alpha = \tan^{-1}\frac{b}{a}$$

6. Rules for solving triangles:
   a. Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
   b. Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$
   c. Area of a triangle: $A = \frac{1}{2}ab\sin C$

Vectors

1. A point with coordinates (*x*, *y*) has a position vector of $\begin{bmatrix} x \\ y \end{bmatrix}$.
2. For points *P* and *Q* with position vectors ***p*** and ***q***, then $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$.
3. The vector equation of a line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + t\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, where ***a*** is a point on the line, ***b*** is the direction of the line, and *t* is a parameter.
4. The cosine of the angle between two vectors is given by:

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{ab} \text{ where } \mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

5. For the scalar product:

$$\mathbf{a}\,.\,\mathbf{b} = \mathbf{b}\,.\,\mathbf{a}$$
$$\mathbf{a}\,.\,(\mathbf{b}+\mathbf{c}) = \mathbf{a}\,.\,\mathbf{b} + \mathbf{a}\,.\,\mathbf{c}$$
$$\mathbf{a}\,.\,\mathbf{b} = 0 \text{ if } \mathbf{a} \text{ is perpendicular to } \mathbf{b},\ \mathbf{a} = 0, \text{ or } \mathbf{b} = 0$$
$$\mathbf{a}\,.\,\mathbf{a} = \mathrm{a}^2$$

6. The vector equation of a plane is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \mu\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, where ***a*** is a point on the line, ***b*** and ***c*** are directions parallel to the plane (but not to one another), and *t* is a parameter.

7. The equation of a plane can also be written as ***n*** . ***r*** = ***n*** . ***a***, where ***a*** is a point on the plane, and ***n*** is the normal vector to the plane.
8. For a plane $ax+by+cz=d$, the normal vector is given by ***n*** = $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
9. In finding angles using planes, take the normal vector to the plane, then calculate the correct angle.

Binomials

1. A binomial expression $(a + b)^n$ can be expanded using the *n*th line of Pascal's triangle:
$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0 b^n$$
2. For $-1 < x < 1$, the binomial series is given as:
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$
3. For calculating errors:
    a. If the measurements are added or subtracted, the errors add.
    b. If a measurement is multiplied by a constant, the error is also multiplied by the constant.
    c. A measurement $a \pm e$ can be written as $a(1 \pm r)$ where *r* is the relative error.
    d. If measurements are multiplied or divided, the relative error of the result is approximately the sum of the relative errors of the measurements, if the relative errors are small.

The Chain Rule

1. To differentiate a function of a function, $y = f(u)$ where $u = f(x)$, then:
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
2. Differentiation by inspection can be applied to integration – if you work out the function that will differentiate approximately to the function you are integrating, you can then scale up your approximate answer with a constant to give the correct solution.
3. Inverse functions can be differentiated using $\frac{dx}{dy} = 1 \div \frac{dy}{dx}$.

Differential Equations

1. A differential equation is one containing a derivative. This gives rise to a family of curves, as if the equation is solved by integration, the constant of integration will always be added. A direction diagram can be used to show this family of curves, by the direction they will take.
2. Differential equations can be solved:
    a. Algebraically – by integrating both sides to remove the differential. This will only work on relatively simple differential equations (that can be integrated).
    b. Numerically – using the step-by-step method from a specified starting point, if the value of *dx* is kept constant, then the value of $\frac{dy}{dx}$ is given by the differential equation and the value of *dy* can be calculated from this. This method will continue until the value of *x* that is needed is found, and it will work on any differential equation.
3. For an equation of the form $\frac{dy}{dx} = \lambda y$, $y = Ae^{\lambda x}$.
4. Differential equations can be formulated by setting up a model to simulate a physical environment, taking certain assumptions into consideration.